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Energetics of a linear array of hollow vortices of finite cross-section

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The energetics of a linear array of hollow or stagnant-cored vortices of finite cross-section in an ideal fluid is studied in this paper. The results are useful in a discussion of the amalgamation of vortex structures in a turbulent mixing layer.

1. Introduction

Baker, Saffman & Sheffield (1976) studied the structure of a linear array of two-dimensional hollow or stagnant-cored vortices of finite cross-section in an ideal fluid. They found that, for a given value of A/L^2, where A is the cross-sectional area and L the uniform spacing between vortices, there were either two or no possible steady states. If $A/L^2 < 0{\cdot}1425$, there are two steady configurations but for hollow vortices the more deformed shape is unstable to infinitesimal two-dimensional disturbances that leave the centres undisplaced; whereas for stagnant-cored vortices the stability is unknown. If $A/L^2 > 0{\cdot}1425$ no steady configurations exist.

They used their results to check the approximate argument of Moore & Saffman (1975) in estimating the maximum value of A/L^2 for which a linear array of two-dimensional uniform vortices of finite cross-section and constant vorticity may exist. Moore & Saffman (1975) modelled the turbulent mixing layer by an array of two-dimensional uniform vortices whose area grows owing to turbulent entrainment of irrotational fluid until A/L^2 is too large for an array to exist. They argued that the vorticity will then be reorganized into a new array of vortices whose spacing and cross-sectional area has been doubled. Thus A/L^2 is reduced by a factor of two so that a new array is possible and the process repeats.

Moore & Saffman (1975) did not test whether this process is consistent with the conservation of energy. The purpose of this paper is to study the energetics of a linear array of hollow or stagnant-cored vortices as a contribution to the understanding of flows with vorticity and to use the results to check the feasibility of the ideas of Moore & Saffman (1975).

2. Calculation of energy difference

The kinetic energy of the fluid in a region of width L, enclosing a member of the array and extending to infinitely large distances from the array, is infinite. However, the energy difference between any two study configurations which have the same

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velocity infinitely far from the array is finite. Thus the energy difference relative to a chosen reference configuration provides a convenient characterization for the energetics of the array.

The plane vortex sheet, which is the limiting case where each vortex is pulled out and flattened, is the most suitable choice for our purposes as a reference configuration, since the results will be used to discuss the validity of a simple model of the turbulent mixing layer. The energy difference is determined by calculating first the energy difference for a finite region of width L and then taking the limit as the region is extended infinitely far.

Consider the array as depicted in figure 1 of Baker *et al.* (1976). For symmetry considerations, we need only calculate the energy of the fluid in the area bounded by $ABCDE$, where A and E are to be considered points of finite distance from the x axis and connected by a streamline passing through both of them. In the limit when A and E are infinitely far from the x axis, the streamline becomes horizontal.

The energy, E, is given by the integral of $\frac{1}{2}\rho(\nabla\phi)^2$ over this area, where ρ is the fluid density and ϕ the velocity potential. Using standard vector analysis the integral may be rewritten as a line integral; therefore

$$E = \tfrac{1}{2}\rho \int \phi(\hat{\mathbf{n}}.\nabla)\,\phi\, ds, \tag{1}$$

where $\hat{\mathbf{n}}$ is a unit normal pointing outward from the area and s is the arclength along the boundary. Since $\phi = 0$ along AB and BC and the normal derivative of ϕ vanishes along the streamlines CD and AE, the only contribution to this integral is along DE. If Γ is the circulation about each vortex,

$$E = -\frac{\rho\Gamma}{8}\int_D^E \frac{\partial\phi}{dx}\,dy = -\frac{\rho\Gamma}{8}\psi_E, \tag{2}$$

where ψ_E is the value of the stream function at E (at $D, \psi = 0$). The relationship between ψ_E and Y, the vertical position of E, can be obtained from equations (3.3) and (3.6) in Baker *et al.* (1976). When Y is large the stream function asymptotes to

$$\psi_E \sim -\frac{\Gamma}{2L}\left\{Y + \frac{L}{2\pi}(1-R^2)\log\left(\frac{1-R^2}{2R}\right) + \frac{L}{2\pi}(1+R^2)\log\left(\frac{1+R^2}{2R}\right)\right\}, \tag{3}$$

where $R = P/2L$ and P is the perimeter of each vortex.

For the plane vortex sheet the energy of the fluid in an area of width L and height $2Y$ is $\rho\Gamma^2 Y/4L$. The energy difference ΔE_s between a steady configuration of stagnant-cored vortices and the plane vortex sheet has the limiting value

$$\Delta E_s = -E - \frac{\rho\Gamma^2 Y}{4L} = \frac{\rho u^2 L^2}{4\pi}\left\{(1-R^2)\log\left(\frac{1-R^2}{2R}\right) + (1+R^2)\log\left(\frac{1+R^2}{2R}\right)\right\}, \tag{4}$$

where the relationship $\Gamma = 2UL$ has been used to express the result in terms of the velocity U at $y = \infty$ and the spacing L. Note that $\Delta E_s \to 0$ as $R \to 1$ as it must, since that is the limit corresponding to the array approaching a plane vortex sheet; $\Delta E \to \infty$ as $R \to 0$, the limit where the vortices become circular. The energy difference is shown as a function of A/L^2 in figure 1.

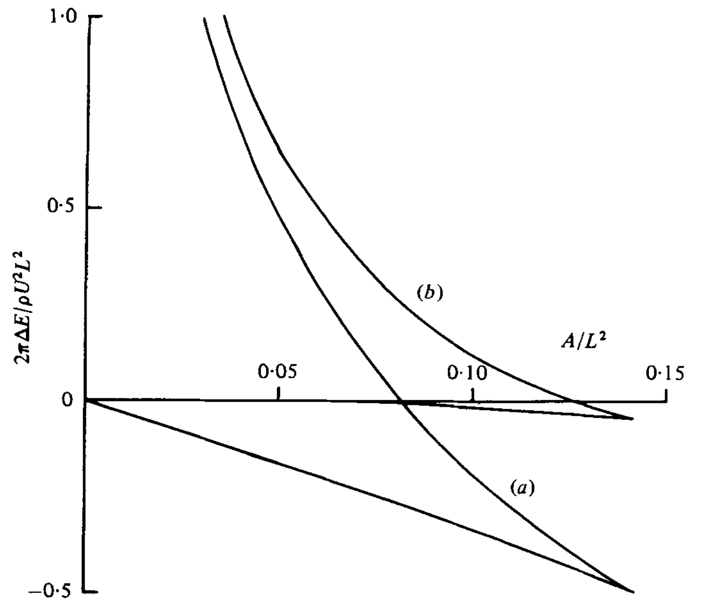

FIGURE 1. The non-dimensional energy difference as a function of the non-dimensional area. (*a*) The stagnant-cored array. (*b*) The hollow array.

If the array is composed of hollow vortices, the value of Y must be shifted to compensate for the area of the vortex. The energy differences, ΔE_H, for a fixed mass of fluid has the limiting value

$$\Delta E_H = \Delta E_S + \tfrac{1}{2}\rho u^2 A. \tag{5}$$

ΔE_H is also shown as a function of A/L^2 in figure 1.

The energy difference has two values for given A/L^2, $A/L^2 < 0{\cdot}1425$, corresponding to the two possible steady configurations. The more deformed shape has lower values of the energy difference. For the hollow array this is the unstable branch of the solutions. The energy difference vanishes at $A/L^2 = 0{\cdot}082$ for stagnant-cored vortices and at $A/L^2 = 0{\cdot}126$ for hollow vortices; the vortices have total height $0{\cdot}34L$, $0{\cdot}28L$ and total width $0{\cdot}38L$, $0{\cdot}56L$ for the two cases, respectively.

3. Discussion

These results may be used to study further the ideas expressed in Moore & Saffman (1975) concerning the behaviour of coherent vortex structures in the turbulent mixing layer. We model the roll-up structures resulting from the Kelvin–Helmholtz instability of the plane vortex sheet by a linear array of stagnant-cored vortices. If no energy is lost during roll-up, the shape of each vortex is given above. Appealing to the hypothesis made by Moore & Saffman (1975) that the coherent vortices grow in size by the turbulent ingestion of irrotational fluid, we see that the stagnant-cored vortices lose energy as their area increases until the critical value $A/L^2 = 0{\cdot}1425$ is reached.

With any further ingestion of irrotational fluid, the array can no longer exist in a steady state. Physically the straining field locally at each vortex structure has become large enough to cause the vortex structures to disintegrate. We follow the suggestion by Moore & Saffman (1975) that a new array forms with the spacing doubled between vortices. If there is no loss of energy during the reorganization of the array, the parameter $2\pi\Gamma E/\rho u^2 L^2$ for the new array is half its previous value. The parameter A/L^2 has the new value 0·106 which is more than half its previous value, indicating that more irrotational fluid must be ingested into the stagnant cores during the reorganization of the array. If energy is lost during the transformation of the array,

$$A/L^2 > 0{\cdot}106.$$

This contradicts the assumption made by Moore & Saffman (1975) that the new vortices have an area which is the sum of the areas of two previous vortices. Aside from this modification, their ideas are still feasible.

Of course, stagnant-cored vortices are not a realistic approximation to the coherent structures in a turbulent mixing layer, so that above arguments are only suggestive. More work is required in understanding the behaviour of arrays of vortex structures before detailed testing of these ideas is possible.

Ferziger (1980) of Stanford University calculated an approximate energy difference for an array of elliptical uniform two-dimensional vortices of constant vorticity. In contrast to the work presented here, he assumes that there is no loss of energy at any stage of the development of the vortex structures and that only during vortex pairing (see Winant & Browand 1974) is irrotational fluid ingested into the vortex structures.

Until further study is made, it is not clear which is the more important process, the growth in area of the vortices leading to their eventual destruction and adsorption into neighbouring vortices or the pairing instability leading to vortex amalgamation. In fact, both processes may be equally important.

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